ROAR NET

Multiobjective Optimisation

1st ROAR-NET Training School

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cost
EUROPEAN COOPERATION IN SCIENCE & TECHNOLOGY

Funded by the European Union

Outline

- Multiobjective Optimization
- Preference Articulation
- Solving Multiobjective Optimisation Problems
- Performance Assessment
- API extension to Multiobjective Optimisation
- Concluding Remarks

Examples

- Book an hotel room
  - Cheapest, most comfortable, closest to the city center

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- Transport for travelling from home to work
  - Cheapest, fastest

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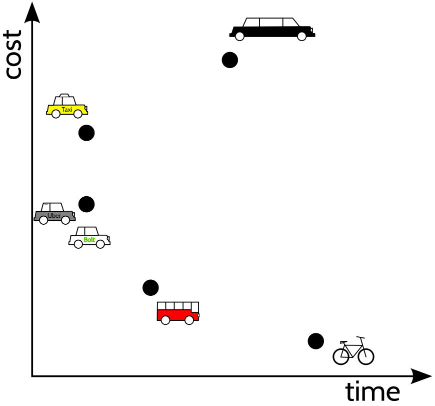
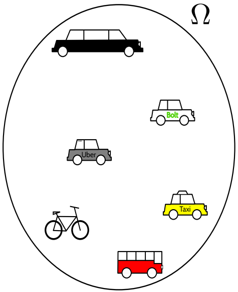


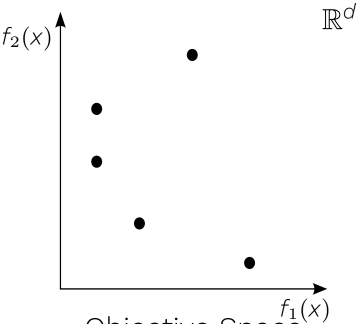
Multiobjective Optimisation

- Considering minimisation:

$$\min_{x \in \Omega} \quad f(x) = (f_1(x), f_2(x), \ldots, f_d(x))$$



Decision Space



Objective Space

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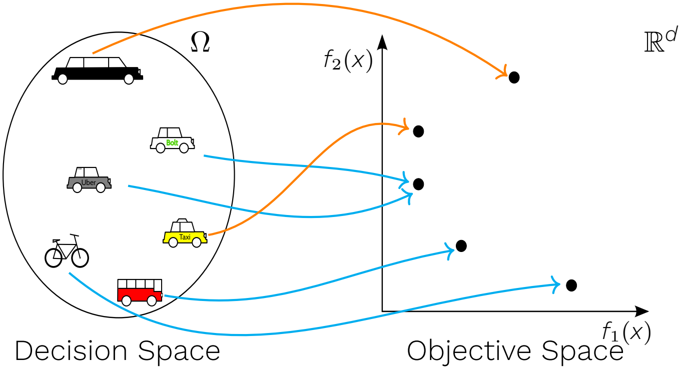

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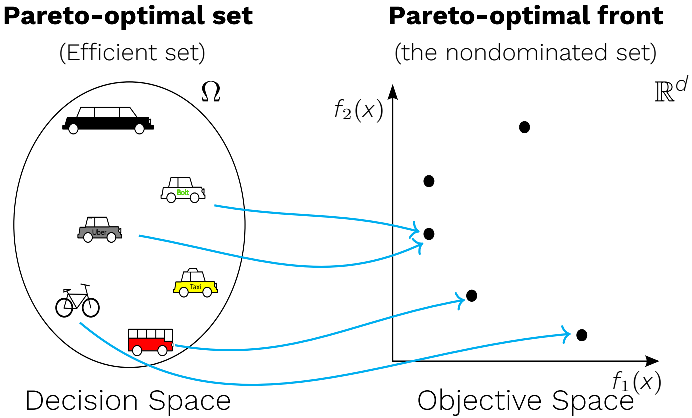
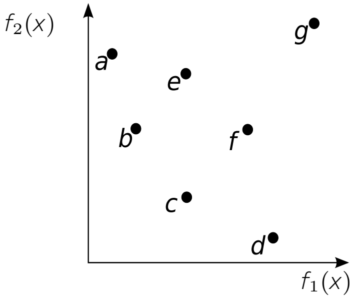


Dominance Relations

Given $u, v \in \mathbb{R}^d$:



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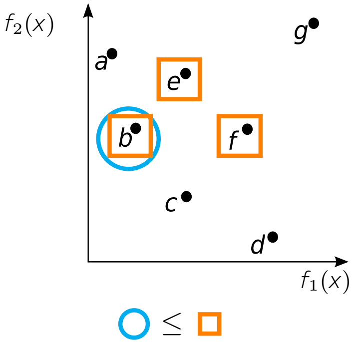

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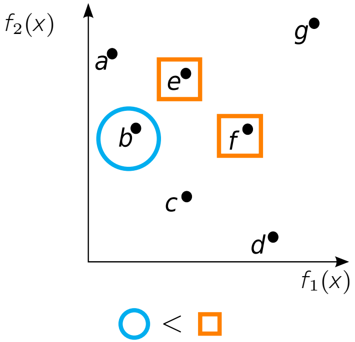
  - If $u \leq v$ and $v \nleq u$



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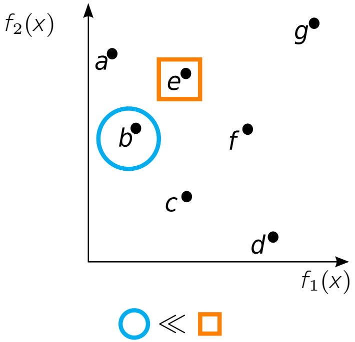

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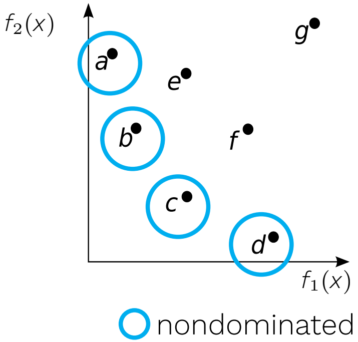
Given $\mathrm{A} \subset \mathbb{R}^d$ and $u \in A$:

- *u is a nondominated point*
  - If there is no $v \in \mathrm{A}$ such that $v < u$



Set-Dominance Relations

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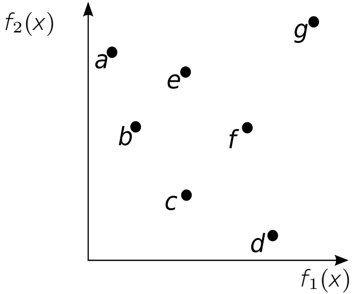

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Given $\mathrm{A}, \mathrm{B} \subset \mathbb{R}^d$:

- A *weakly dominates* B $(\mathrm{A} \preceq \mathrm{B})$
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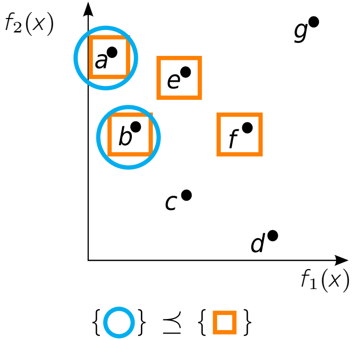

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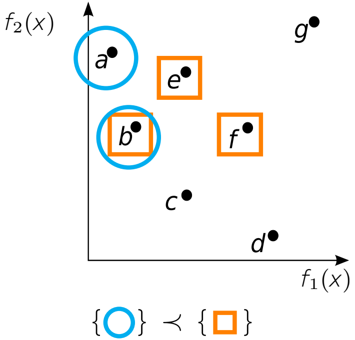

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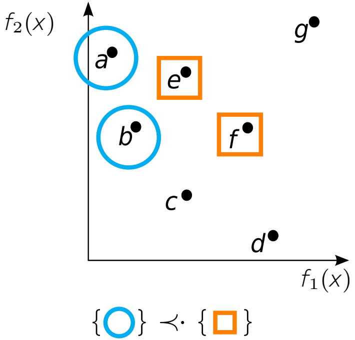

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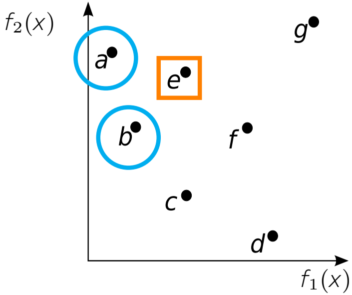
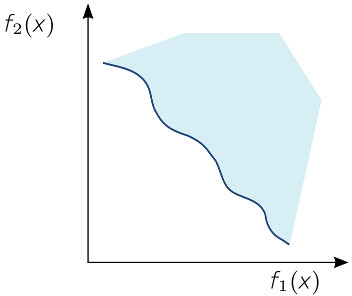


$\{\bigcirc\} \prec\!\!\prec \{\square\}$

Decision Making

- Conflicting objectives
  - Pareto-optimal front
    - Usually not known
    - Possible infinite



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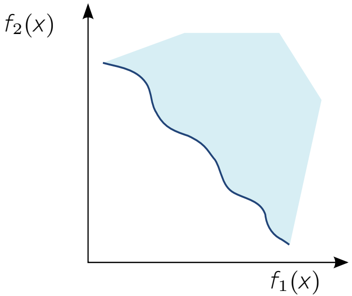
- The best solution depends on the Decision Maker (DM) preferences
  - Subjective information



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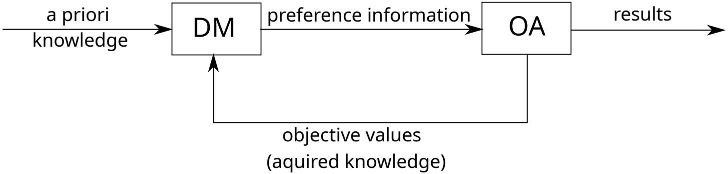
     - Focus on solutions on the preferable regions

Preference Articulation as a Process

- The optimisation process does not have to be static
- Preference articulation can be viewed as a process



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    - Does it provide an approximation guarantee?
  - Provide bounds on the location of the Pareto front

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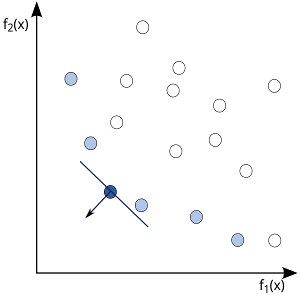
$$\min_{x \in \Omega} \sum_{i=1}^{d} w_i f_i(x)$$



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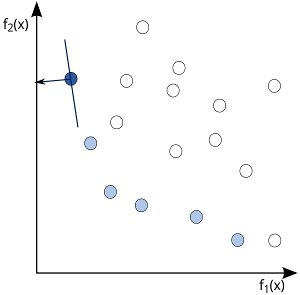

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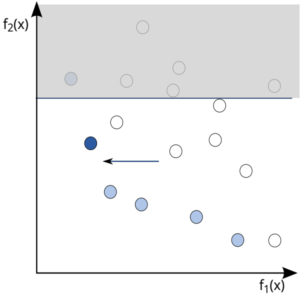
$$\min_{x \in \Omega} \sum_{i=1}^{d} w_i f_i(x)$$

  - ε-constraint scalarisation

$$\min_{x \in \Omega} f_1(x)$$
$$\text{s.t.} \quad f_i(x) \leqslant \varepsilon_i, \; i = 2, \ldots, d$$



Scalarisations

- Transform the multiobjective problem into a (parameterised) single-objective problem
- Solve multiple times with different scalarisation parameters
- Examples of scalarisations
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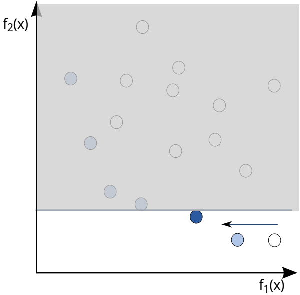

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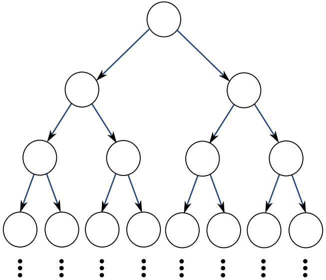
  - Completeness: Every point in the Pareto front can be determined by an appropriate choice of the parameters

Multiobjective Branch-and-Bound

- Branching

- Bounding



Multiobjective Branch-and-Bound

- Branching
  - Branching strategy (e.g., choose next branching variable x_i)

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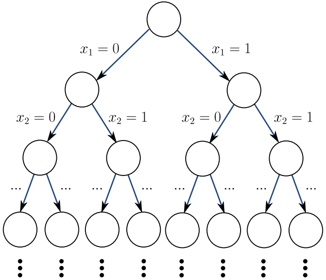

Multiobjective Branch-and-Bound

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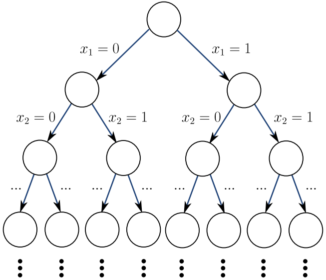

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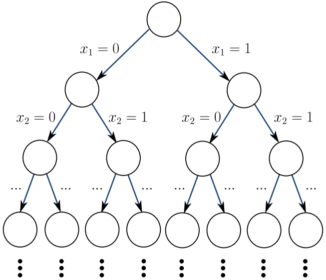

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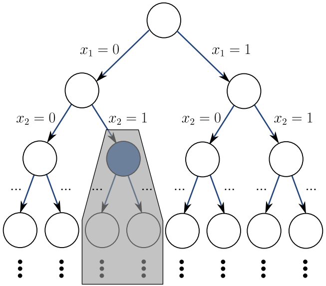

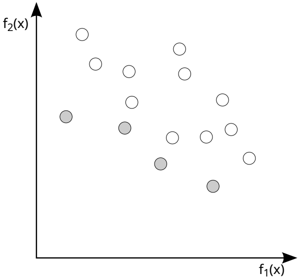

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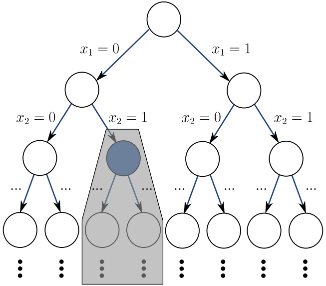

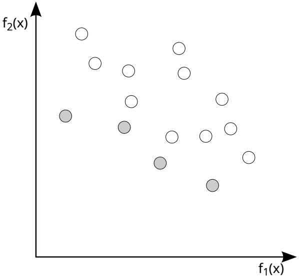

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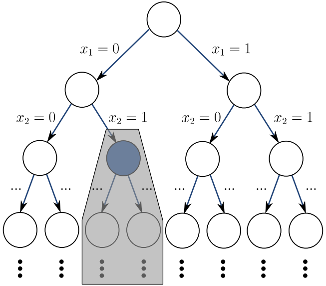

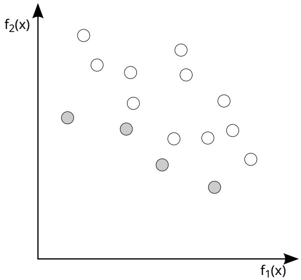

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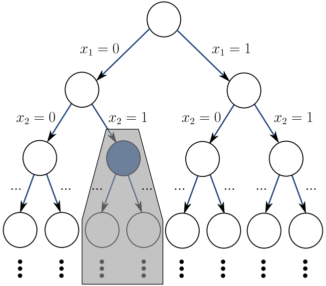

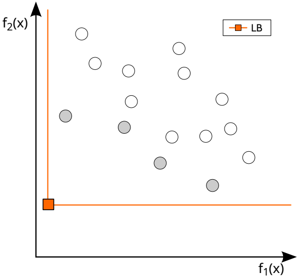

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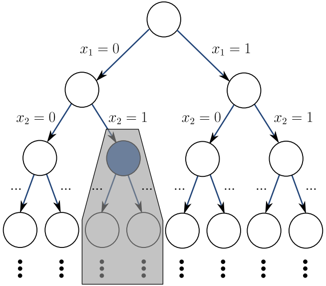

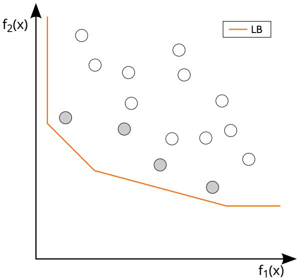

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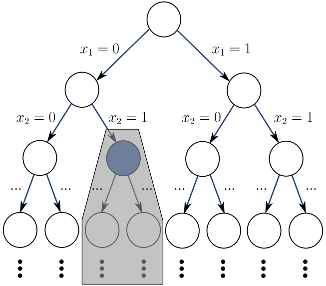

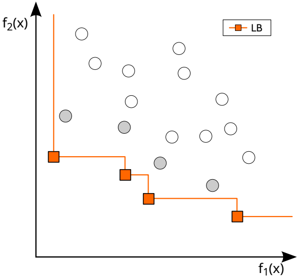

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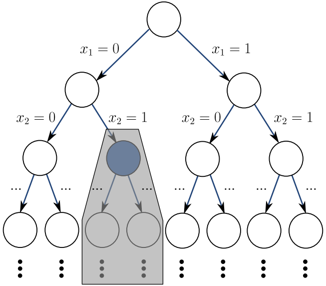

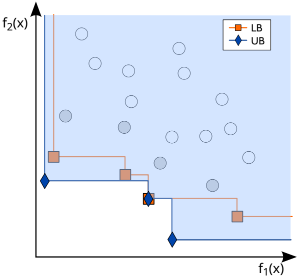

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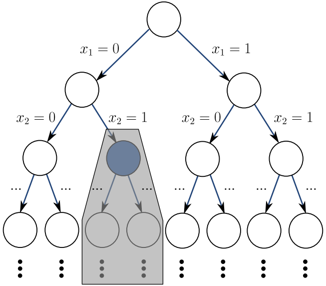

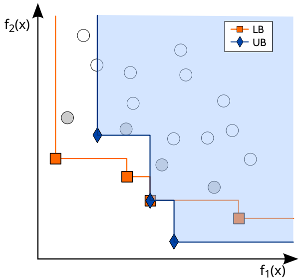

Evolutionary Multiobjective Optimisation (EMO)

- Evolutionary Algorithms (EAs)
  - Inspired on natural selection
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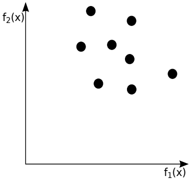

Initial population (generation 0)

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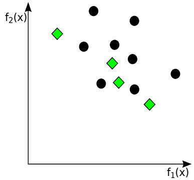

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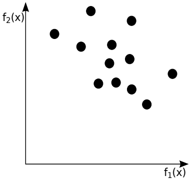

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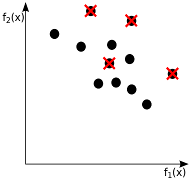

(Environmental) selection:
Discard solutions

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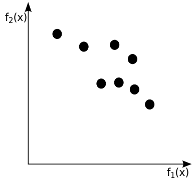

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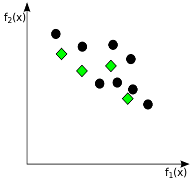

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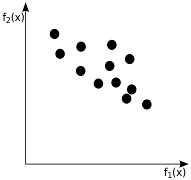

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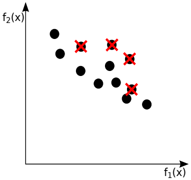

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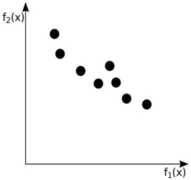

Population (generation 2)

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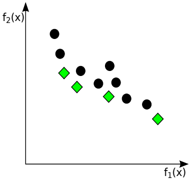

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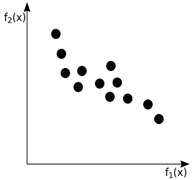

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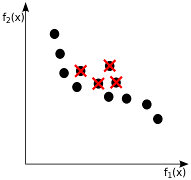

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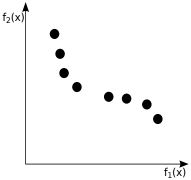

Population (generation 3)

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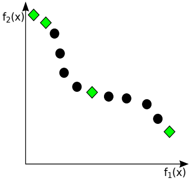

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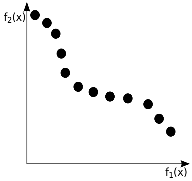

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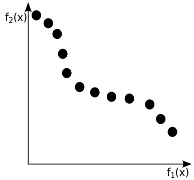

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  - Discriminate incomparable solutions
  - When not available
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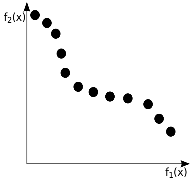

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Selection in EMO

- Crucial to the EA’s ability to approximate the Pareto front as a whole

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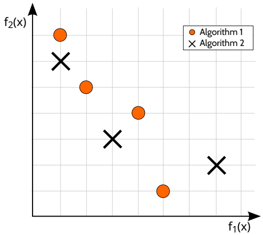
    - Based on (set-)quality indicators
    - Given a set of n solutions find the subset of k solutions that maximizes a given quality indicator

Performance Assessment

- Different optimisation algorithms usually produce different results



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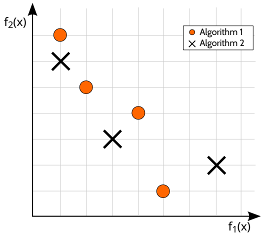

Performance Assessment

- Different optimisation algorithms usually produce different results
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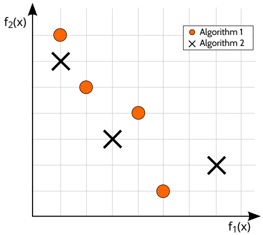
- The solutions produced in one optimisation run may be incomparable to those produced in another run



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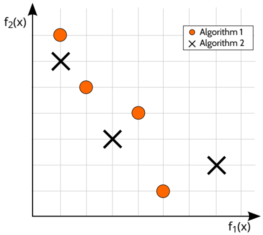

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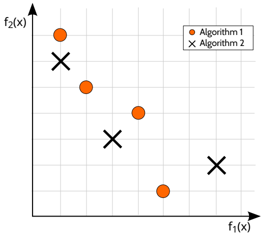

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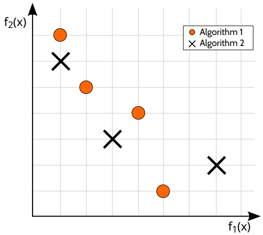

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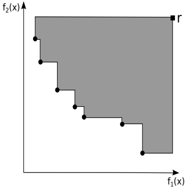

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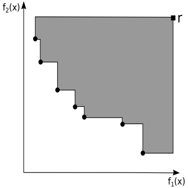

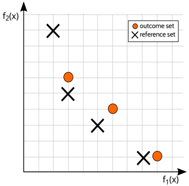

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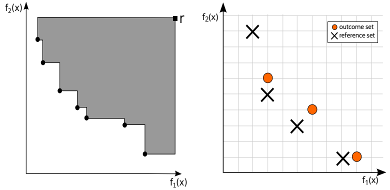

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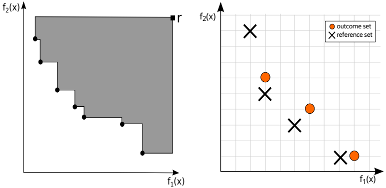
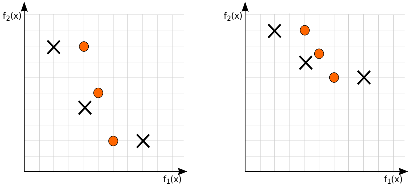


- Performance assessment
- Guide the search process (subset selection view)
  - Find a subset of k solutions that maximizes a given quality indicator

Properties of Quality Indicators

Scaling invariance/independence



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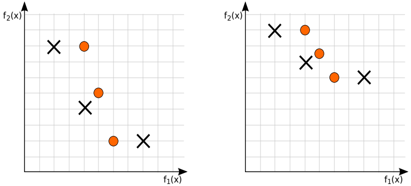

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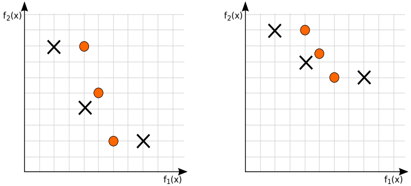

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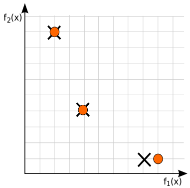

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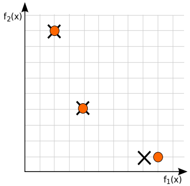

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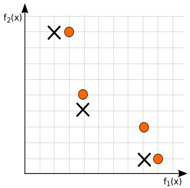

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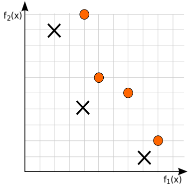

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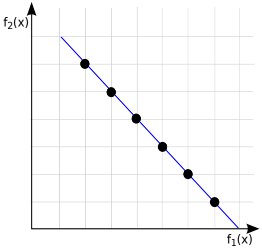

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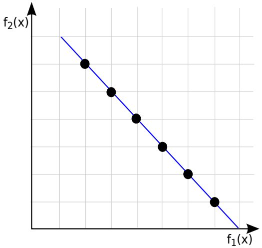

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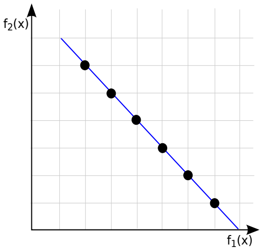

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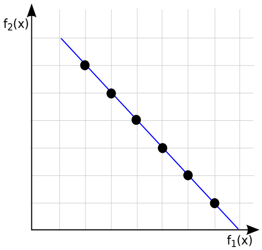

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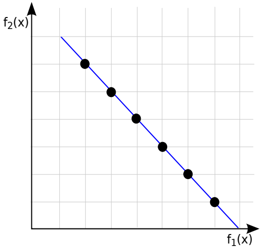

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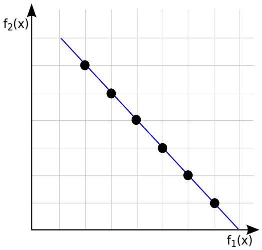

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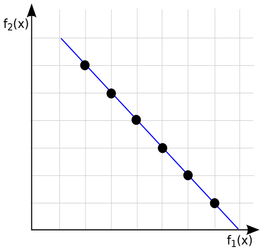

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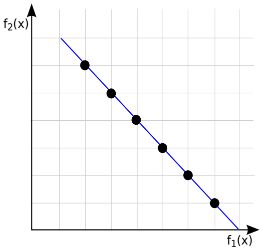

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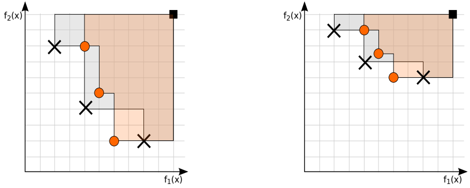

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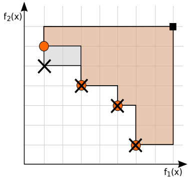

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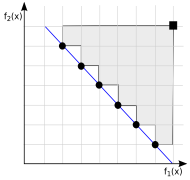
  - Ex.: The points in an optimal subset of a two-dimensional continuous linear Pareto front are evenly spaced



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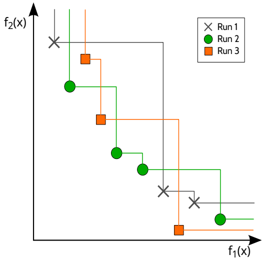

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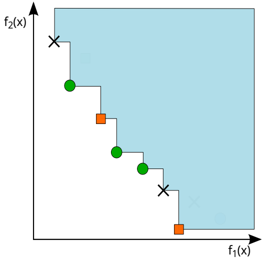

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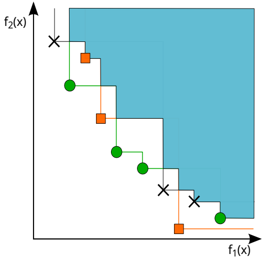
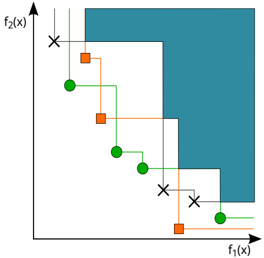
- Example with 3 runs
  - Region of the space attained by at least $\frac{2}{3}$ of the runs



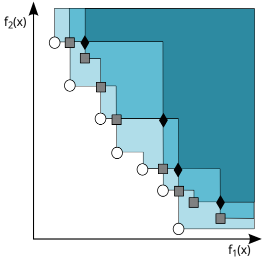
Empirical Attainment Function

- Allows the distribution of the outcomes of different runs of an optimisation algorithm to be studied in terms of location
- Example with 3 runs
    - Region of the space attained by at least $\frac{3}{3}$ of the runs



Empirical Attainment Function

- Allows the distribution of the outcomes of different runs of an optimisation algorithm to be studied in terms of location
- Example with 3 runs
  - All attainment regions



API extension to Multiobjetive Optimisation

Only affects solution evaluation:

```
objective_value(Solution) : double[0..1]
objective_value_increment(Move, Solution) : double[0..1]
```

Considers new abstract types `Value` and `Increment`:

```
objective_value(Solution) : Value[0..1]
objective_value_increment(Move, Solution) : Increment[0..1]
```

API extension to Multiobjective Optimisation

New type: `PreferenceModel`

Scalarisations:

```
scalarisation(PreferenceModel, Value): real
```

Preference relations:

```
better_or_indifferent(PreferenceModel, Value, Value): boolean
better(PreferenceModel, Value, Value): boolean
indifferent(PreferenceModel, Value, Value): boolean
incomparable(PreferenceModel, Value, Value): boolean

compare(PreferenceModel, Value, Value):
                {Better | Worse | Indifferent}
compare_partial(PreferenceModel, Value, Value):
                {Better | Worse | Indifferent | Incomparable}
```

API extension to Multiobjetive Optimisation

Other non-scalarising approaches

```
selection(PreferenceModel, Value[0..*], n=None) : int[0..*]
ranking(PreferenceModel, Value[0..*]) : int[0..*]
```

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    - Biases towards (sets of) solutions
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  - The theoretical properties of quality indicators and their implications
- Leads to better usage of optimisation algorithms and performance assessment tools
- The API allows to incorporate different models of preferences

Thanks!

Acknowledgments

This presentation is based upon work from COST Action Randomised Optimisation Algorithms Research Network (ROAR-NET), CA22137, supported by COST (European Cooperation in Science and Technology).

This work is financed by National Funds through the Portuguese funding agency, FCT - Fundação para a Ciência e a Tecnologia within project: 2022.08367.CEECIND/CP1717/CT0001

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Funded by the European Union

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